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On Vector Implicit Variational Inequalities and Complementarity Problems*

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Abstract. In this paper, two vector implicit variational inequalities and three vector implicit complementarity problems are introduced with a more general order in ordered Banach spaces and the equivalence between them is studied under certain assumptions. Furthermore, some existence theorems of the vector implicit variational inequalities are proved. We answered the open question proposed by Rapcsák [Nonconvex Optim., Appl., Vol. 38, Kluwer Acad. Publ., Dordrecht, 2000, pp. 371–380].

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1. Introduction

A vector variational inequality in a finite-dimensional Euclidean space was introduced first by Giannessi [3]. This is a generalization of a scalar variational inequality to the vector case by virtue of multi-criteria consideration. Throughout over last 20 years development, existence theorems of solutions of vector variational inequalities in various situations have been studied by many authors (see, for example, [1, 2, 4, 5, 7–10] and the references therein). At the same time, vector variational inequality has found many of its applications in vector optimization [9], approximate vector optimization [1], vector equilibria [4, 5] and vector traffic equilibria [10].

In 1990, Chen and Yang [2] defined a vector variational inequality and three vector complementarity problems, i.e., the weak, positive and strong vector complementarity in ordered Banach spaces and proved the existence theorem for them. And, the equivalence between them was established under some additional assumptions. In 1993, Yang [8] analyzed the

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relations among vector variational inequality, vector complementarity, minimal element problems, and fixed point problems.

Recently, Rapcsák [7] introduced the weak order in Banach spaces. Based on this new order, the author of [7] introduced a vector variational inequality and three vector complementarity problems and discussed some relations between them. At the end of the paper [7], Rapcsák proposed an open question, i.e., in the case of ordering (2.1), the existence of a solution to (VVIP) or (WVCP) (see Section 3).

In this paper, two vector implicit variational inequalities and three vector implicit complementarity problems, i.e., the weak, positive and strong vector implicit complementarity are introduced with a more general order in ordered Banach spaces and the equivalence between them is established under certain conditions. Furthermore, some existence theorems of two vector implicit variational inequalities are proved. We answered the open question proposed by Rapcsák [7].

2. Preliminaries

Let Y be a real Banach space. A nonempty subset P of Y is said to be a convex cone if (i)$P+P=P$; (ii)$\lambda P \subseteq P$ for all $\lambda > 0$. A cone P is called pointed if P is convex and $P \cap \{-P\} = \{0\}$. An ordered Banach space (Y, P) is a real Banach space Y with an order defined by a closed convexcone $P \subseteq Y$ with apex at the origin, in the form of

$$y \leqslant_P x \Leftrightarrow x - y \in P, \quad \forall x, y \in Y$$

and

$$y \nleqslant_P x \Leftrightarrow x - y \notin P, \quad \forall x, y \in Y.$$

If the interior of P, say int$P$, is nonempty, then a weak order in Y is also defined by

$$y \nleqslant_{\mathrm{int}P} x \Leftrightarrow x - y \notin \mathrm{int}P, \quad \forall x, y \in Y.$$

Remark that

$$y \leqslant_P x \Leftrightarrow x \geqslant_P y$$

and

$$y \nleqslant_P x \Leftrightarrow x \ngeqslant_P y, \qquad y \nleqslant_{\mathrm{int}P} x \Leftrightarrow x \ngeqslant_{\mathrm{int}P} y.$$

Let (X, K) and (Y, P) be two ordered Banach spaces with int$P \neq \emptyset$. Denote by $L(X, Y)$ the set of all linear continuous mapping from X to Y. The weak (int$P$)-dual cone $K^{w^+}_{\mathrm{int}P}$ of K is defined by

$$K^{w^+}_{\mathrm{int}P} = \{q \in L(X, Y) : \langle q, x \rangle \nleqslant_{\mathrm{int}P} 0, \quad \forall x \in K\},$$

where the subscript int$P$ means that the weak order $\not\leqslant_{\text{int}P}$ is defined by int$P$, and the value of $q \in L(X, Y)$ at $x \in K$ is denoted by $\langle q, x\rangle$. The strong (P)-dual cone $K_P^{s^+}$ of K is defined by

$$K_P^{s^+} = \{q \in L(X, Y) : 0 \leqslant_P \langle q, x\rangle, \ \forall x \in K\}.$$

It is obviously that $K_{\text{int}P}^{w^+}$ and $K_P^{s^+}$ are nonempty, since the null linear mapping belongs to $K_{\text{int}P}^{w^+}$ and $K_P^{s^+}$. It is easy to see that $K_P^{s^+} \subseteq K_{\text{int}P}^{w^+}$ if P is pointed. If $Y = R$ and $P = R_+$, then the weak (int$P$) and strong (P)-dual cones of K are reduced to the usual dual cone K^* of K given by

$$K_{\text{int}P}^{w^+} = K_P^{s^+} = K^* = \{q \in L(X, Y) : 0 \leqslant \langle q, x\rangle, \ \forall x \in K\}.$$

Let $D \subseteq Y$ be a convex cone. In [7], Rapcsák introduced the following order

$$y \not\leqslant_D x \Leftrightarrow x - y \notin D, \quad \forall x, y \in Y. \tag{2.1}$$

Since cl$D$ is a closed convex cone,

$$y \not\leqslant_{\text{cl}D} x \Leftrightarrow x - y \in \text{cl}D, \quad \forall x, y \in Y$$

and

$$y \not\leqslant_{\text{cl}D} x \Leftrightarrow x - y \notin \text{cl}D, \quad \forall x, y \in Y.$$

where cl$D$ denotes the closure of D. Note that

$$y \not\leqslant_D x \Leftrightarrow x \not\geqslant_D y, \quad y \leqslant_{\text{cl}D} x \Leftrightarrow x \geqslant_{\text{cl}D} y$$

and

$$y \not\leqslant_{\text{cl}D} x \Leftrightarrow x \geqslant_{\text{cl}D} y.$$

This order satisfies the following properties:

$$y \not\leqslant_D x \Leftrightarrow y + w \not\leqslant_{D\, x+w}, \quad \forall x, y, w \in Y;$$
$$y \not\leqslant_D x \Leftrightarrow \lambda y \not\leqslant_D, \lambda x, \quad \forall x, y \in Y \text{ and } \lambda > 0.$$

Similarly, a weak (D)-dual $K_D^{w^+}$ of K is defined by

$$K_D^{w^+} = \{q \in L(X, Y) : \langle q, x\rangle \not\leqslant_D 0, \ \forall x \in K\}.$$

A strong (cl$D$)-dual cone $K_{\text{cl}D}^{s^+}$ of K is defined by

$$K_{\text{cl}D}^{s^+} = \{q \in L(X, Y) : 0 \leqslant_{\text{cl}D} \langle q, x\rangle, \ \forall x \in K\},$$

where cl$D$ denotes the closure of D. Moreover, Rapcsák introduced the following binary relation:

$$y =_D 0, \quad y \in Y \Leftrightarrow \begin{cases} y \in \text{frt}(\text{cl}D) \cup \text{frt}(-\text{cl}D) \setminus -D \\ \text{and} \\ -y \in \text{frt}(\text{cl}D) \cup \text{frt}(-\text{cl}D) \setminus -D, \end{cases} \tag{2.2}$$

where frt$A$ denotes the frontier of A. By this relation, the set of zero points with respect to a convex cone D is nonempty if frt(cl$D$)$\backslash D$ is nonempty. we note that

$$y =_D 0, \quad y \in Y \Leftrightarrow \lambda_y =_D 0, \quad y \in Y, \ \lambda > 0.$$

3. Vector Implicit Variational Inequalities and Complementarity Problems

Let (X, K) and (Y, P) be two ordered Banach spaces with int$P \neq \emptyset$, and D a convex cone in Y. Let $f : X \to L(X, Y)$. In [7], Rapcsák considered one vector variational inequality problem and three vector complementary problems as follows:

1. Vector variational inequality problem (in short VVIP): find $x^* \in K$ such that $\langle f(x^*), y - x^* \rangle \nleq_D 0, \ \forall y \in K$.
2. Weak vector complementary problem (in short WVCP): find $x^* \in K$ such that $f(x^*) \in K_D^{w^+}$ and $\langle f(x^*), x^* \rangle =_D 0$.
3. Positive vector complementary problem (in short PVCP): find $x^* \in K$ such that $f(x^*) \in K_{\mathrm{cl}D}^{s^+}$ and $\langle f(x^*), x^* \rangle =_D 0$.
4. Strong vector complementary problem (in short SVCP): find $x^* \in K$ such that $f(x^*) \in K_{\mathrm{cl}D}^{s^+}$ and $\langle f(x^*), x^* \rangle =_D 0$.

At the end of the paper [7], Rapcsák proposed an open question, i.e., in the case of ordering (2.1), the existence of a solution to (VVIP) or (WVCP).

Let $g : K \to K$ be a mapping. In this paper, we consider two kinds of vector implicit variational inequality problems and three kinds of vector implicit complementary problems as follows:

1. Weak vector implicit variational inequality problem (in short WVIVIP): find $x^* \in K$ such that $\langle f(x^*), y - g(x^*) \rangle \nleq_D 0, \ \forall y \in K$.
2. Strong vector implicit variational inequality problem (in short SVIVIP): find $x^* \in K$ such that $\langle f(x^*), y - g(x^*) \rangle \nleq C_D 0, \ \forall y \in K$.
3. Weak vector implicit complementary problem (in short WVICP): find $x^* \in K$ such that $f(x^*) \in K_{\mathrm{cl}D}^{w^+}$ and $\langle f(x^*), g(x^*) \rangle =_D 0$.
4. Positive vector implicit complementary problem (in short PVICP): find $x^* \in K$ such that $f(x^*) \in K_{\mathrm{cl}D}^{s^+}$ and $\langle f(x^*), g(x^*) \rangle =_D 0$.
5. Strong vector implicit complementary problem (in short SVICP): find $x^* \in K$ such that $f(x^*) \in K_{\mathrm{cl}D}^{s^+}$ and $\langle f(x^*), g(x^*) \rangle = 0$.

REMARK 3.1. If g is an identify mapping on K, then (WVIVIP), (WVICP), (PVICP) and (SVICP) reduce to (VVIP), (WVCP), (PVCP) and (SVCP), respectively, which have been studied by Rapcsák in [7]. Furthermore, if D is an open convex cone, then (WVIVIP), (WVICP), (PVICP), and (SVICP) reduce to (VVIP), (WVCP), (PVCP) and (SVCP), respectively, which have been studied by Chen and Yang [2], and by Yang [8].

DEFINITION 3.1 [7]. A convex cone D is acute if cl$D$ is pointed. A convex cone D is correct if

$$\mathrm{cl}D + D \backslash (D \cap -D) \subseteq D.$$

LEMMA 3.1 [7]. *If $D \subseteq Y$ is a convex cone, then*

(1) $0 \nleqslant_D y$ *and* $x \leqslant_D y$ *imply that* $0 \nleqslant_D x, x, y \in Y$;
(2) $y \nleqslant_D 0$ *and* $y \leqslant_D x$ *imply that* $x \nleqslant_D 0, x, y \in Y$.

LEMMA 3.2. *If $D \subseteq Y$ is an acute cone and $0 \notin -D$, then*

(1) $y \geqslant_{\mathrm{cl}D} 0$ *implies that* $y \nleqslant_D 0, \ \forall y \in Y$;
(2) $K^{s^+}_{\mathrm{cl}D} \subseteq K^{w^+}_D$.

Proof. (1) It is easy to see that $y \geqslant_{\mathrm{cl}D} 0 \Leftrightarrow y \in \mathrm{cl}D$. Since D is an acute cone and $0 \notin -D$, we know that $y \notin -D$, which is equivalent to $y \nleqslant_D 0$.

(2) Let $q \in K^{s^+}_{\mathrm{cl}D}$. Then

$$q \in L(X, Y) \quad \text{and} \quad 0 \leqslant_{\mathrm{cl}D} \langle q, x \rangle, \quad \forall x \in K.$$

Since D is an acute cone and $0 \notin -D$, it follows from conclusion (1) that

$$q \in L(X, Y) \quad \text{and} \quad \langle q, x \rangle, \nleqslant_D 0, \quad \forall x \in K$$

and so $q \in K^{w^+}_D$. This completes the proof. □

THEOREM 3.1. *Let $D \subseteq Y$ be an acute convex cone with apex at the origin.*

(i) If x^* solves (SVIVIP) and $0 \notin -D$, then x^* solves (WVIVIP).
(ii) If x^* solves (WVIVIP) and $\langle f(x^*), g(x^*) \rangle \in \mathrm{frt}(\mathrm{cl}D) \cup \mathrm{frt}(-\mathrm{cl}D)$, then x^* solves (WVICP).
(iii) If x^* solves (WVICP), $\langle f(x^*), g(x^*) \rangle \in \mathrm{frt}(-\mathrm{cl}D) \backslash -D, 0 \notin -D$ and $-D$ is correct, then x^* solves (WVIVIP).
(iv) If x^* solves (PVICP) and $0 \notin -D$, then x^* solves (WVICP).
(v) If x^* solves (SVICP) and $0 \notin -D$, then x^* solves (PVICP).
(vi) If x^* solves (SVICP) and $0 \notin -D$, then x^* solves (WVIVIP).

Proof. (i) Let $x^* \in K$ be a solution of (SVIVIP), i.e., $x^* \in K$ such that

$$\langle f(x^*), y - g(x^*) \rangle \geqslant_{\mathrm{cl}D} 0, \quad \forall y \in K.$$

Since D is an acute and $0 \notin -D$, from Lemma 3.2, we have

$$\langle f(x^*), y - g(x^*) \rangle \nleqslant_D 0, \quad \forall y \in K,$$

which gives that x^* is a solution of (WVIVIP).

(ii) Let $x^* \in K$ be a solution of (WVIVIP), i.e., $x^* \in K$ such that

$$\langle f(x^*), y - g(x^*) \rangle \nleqslant_D 0, \quad \forall y \in K. \tag{3.1}$$

Let $y = x + g(x^*)$ for all $x \in K$. Since K is a convex cone, we know that $y \in K$ and thus

$$\langle f(x^*), x\rangle \not\leq_D 0,$$

which implies that $f(x^*) \in K_D^{w^+}$.

Since $y = (1 + \lambda)g(x^*)$, $\lambda > -1$, belong to K, it follows (3.1) that

$$\lambda\langle f(x^*), g(x^*)\rangle = \langle f(x^*), \lambda g(x^*)\rangle, \not\leq_D 0, \quad \lambda \in (-1, +\infty),$$

from which, we obtain

$$\langle f(x^*), g(x^*)\rangle \notin D \quad \text{and} \quad \langle f(x^*), g(x^*)\rangle \notin -D. \tag{3.2}$$

By the assumption that $\langle f(x^*), g(x^*)\rangle \in \mathrm{frt}(\mathrm{cl}D) \cup \mathrm{frt}(-\mathrm{cl}D)$, we have $-\langle f(x^*), g(x^*)\rangle \in \mathrm{frt}(\mathrm{cl}D) \cup \mathrm{frt}(-\mathrm{cl}D)$. It follows from (2.2) and (3.2) that

$$\langle f(x^*), g(x^*)\rangle =_D 0,$$

which shows that x^* is a solution of (WVICP).

(iii) Let $x^* \in K$ be a solution of (WVICP), i.e., $x^* \in K$ such that

$$f(x^*) \in K_D^{w^+} \qquad \text{and} \qquad \langle f(x^*), g(x^*)\rangle =_D 0. \tag{3.3}$$

Since $f(x^*) \in L(X, Y)$ and $K \subset X$, it is easy to see that

$$\langle f(x^*), y - g(x^*)\rangle = \langle f(x^*), y\rangle - \langle f(x^*), g(x^*)\rangle, \quad \forall y \in K \tag{3.4}$$

and from assumption, we obtain

$$0 \leqslant_{\mathrm{cl}D} -\langle f(x^*), g(x^*)\rangle. \tag{3.5}$$

If $\langle f(x^*), g(x^*)\rangle = 0$, then it follows from (3.3) and (3.4) that the conclusion holds. Suppose that $\langle f(x^*), g(x^*)\rangle \neq 0$, and

$$\langle f(x^*), y\rangle \not\leq_{\mathrm{cl}D} 0, \quad \forall y \in K. \tag{3.6}$$

Then, from (3.5), we obtain

$$\langle f(x^*), y\rangle \leqslant_{\mathrm{cl}D} \quad \langle f(x^*), y - g(x^*)\rangle. \tag{3.7}$$

Thus, by Lemma (3.1), (3.6) and (3.7) imply that

$$\langle f(x^*), y - g(x^*)\rangle \not\leq_{\mathrm{cl}D} 0, \quad \forall y \in K. \tag{3.8}$$

A consequence of (3.8) is that

$$\langle f(x^*), y - g(x^*)\rangle \not\leq_D 0, \quad \forall y \in K,$$

i.e., x^* is a solution of (WVIVIP). If $\langle f(x^*), g(x^*)\rangle \neq 0$ and there exists $y \in K$ such that $\langle f(x^*), y\rangle \in -\mathrm{cl}D \setminus -D$ and $\langle f(x^*), y - g(x^*)\rangle \in -D$, then from assumptions $\langle f(x^*), g(x^*)\rangle \in \mathrm{frt}(-\mathrm{cl}D)\setminus -D$, $0 \notin -D$ (i.e., $D \cap -D = \emptyset$), and the correctness of the cone $-D$, we have

$$\langle f(x^*), y\rangle = \langle f(x^*), y - g(x^*)\rangle + \langle f(x^*), g(x^*)\rangle \in -D - \mathrm{cl}D \subseteq -D,$$

which is a contradiction. If there exists $y \in K$ such that $\langle f(x^*), y \rangle \in -D$, then $\langle f(x^*), y \rangle \leqslant_D 0$, and this contradicts with $f(x^*) \in K_D^{w^+}$.

(iv) Let $x^* \in K$ be a solution of (PVICP), i.e., $x^* \in K$ such that

$$f(x^*) \in K_{\mathrm{cl}D}^{s^+} \quad \text{and} \quad \langle f(x^*), g(x^*) \rangle =_D 0.$$

Since D is an acute and $0 \notin -D$, then it follows from Lemma 3.2 that

$$f(x^*) \in K_D^{w^+},$$

which shows that x^* is also a solution of (WVICP).

(v) Let $x^* \in K$ be a solution of (SVICP), i.e., $x^* \in K$ such that

$$f(x^*) \in K_{\mathrm{cl}D}^{s^+} \quad \text{and} \quad \langle f(x^*), g(x^*) \rangle = 0.$$

Note that D is an acute and $0 \notin -D$, then it follows from relation (2.2) that

$$\langle f(x^*), g(x^*) \rangle =_D 0,$$

which shows that x^* is also a solution of (PVICP).

(vi) Let $x^* \in K$ be a solution of (SVICP), i.e., $x^* \in K$ such that

$$f(x^*) \in K_{\mathrm{cl}D}^{s^+} \quad \text{and} \quad \langle f(x^*), g(x^*) \rangle = 0.$$

Since D is an acute and $0 \notin -D$, then by Lemma 3.2, we have that

$$\langle f(x^*), y - g(x^*) \rangle = \langle f(x^*), y \rangle - \langle f(x^*), g(x^*) \rangle \not\leqslant_D 0,$$

which prove that x^* is also a solution of (WVIVIP). This completes the proof. □

REMARK 3.2. If g is an identity mapping on K, then from Theorem 3.1, we have

(i) if x^* solves (SVVIP) and $0 \notin -D$, then x^* solves (WVVIP);
(ii) if x^* solves (WVVIP) and $\langle f(x^*), x^* \rangle \in \mathrm{frt}(\mathrm{cl}D) \cup \mathrm{frt}(-\mathrm{cl}D)$, then x^* solves (WVCP);
(iii) if x^* solves (WVCP), $\langle f(x^*), x^* \rangle \in \mathrm{frt}(-\mathrm{cl}D) \setminus -D, 0 \notin -D$ and $-D$ is correct, then x^* solves (WVVIP);
(iv) if x^* solves (PVCP) and $0 \notin -D$, then x^* solves (WVCP);
(v) if x^* solves (SVCP) and $0 \notin -D$, then x^* solves (PVCP);
(vi) if x^* solves (SVCP) and $0 \notin -D$, then x^* solves (WVVIP).

REMARK 3.3. The following example shows that the assumption $\langle f(x^*), x^* \rangle \in \mathrm{frt}(\mathrm{cl}D) \cup \mathrm{frt}(-\mathrm{cl}D)$ in (ii) of Corollary 3.1 is necessary.

EXAMPLE 3.1. Let $X = R, Y = R^2$, $K = [0, +\infty)$ and $D = (0, \infty) \times (0, \infty)$. Let

$$f(x) = (x, -x), \qquad \langle f(x), z \rangle = (zx, -zx)$$

for any $x, z \in K$. Then,

$$\langle f(x), y - x \rangle = (x(y - x), -x(y - x))$$

for any $x, y \in K$. Since $D \subseteq Y$ is an acute convex cone with apex at the origin, then $x^* = 1 \in K$ is a solution of (WVVIP). However, $\langle f(x^*), x^* \rangle = (1, -1) \notin \mathrm{frt}(\mathrm{cl}D) \cup \mathrm{frt}(-\mathrm{cl}D)$. It is easy to verify that $\langle f(x^*), y \rangle = (y, -y) \notin -D$ for any $y \in K$, i.e., $f(x^*) \in K_D^{w^+}$. Since $\pm\langle f(x^*), (x^*) \rangle = \pm(1, -1) \notin \mathrm{frt}(\mathrm{cl}D) \cup \mathrm{frt}(-\mathrm{cl}D)$, then $\langle f(x^*), x^* \rangle =_D 0$ does not hold because of binary relation (2.2). That is to say, $x^* = 1 \in K$ does not solve (WVCP).

LEMMA 3.3 [6]. *Let K be a nonempty, convex subset of a Hausdorff topological vector space X, and A be a nonempty subset of $K \times K$. Suppose the following assumptions hold:*

(i) for each $x \in K$, $(x, x) \in A$;
(ii) for each $y \in K$, $A_y = \{x \in K : (x, y) \in A\}$ is closed in K;
(iii) for each $x \in K$, $A_x = \{y \in K : (x, y) \notin A\}$ is convex or empty;
(iv) there exists a nonempty compact convex subset C of K such that $B = \{x \in K : (x, y) \in A, \forall y \in C\}$ is compact in K.

Then there exists $x^* \in K$ such that $\{x^*\} \times K \subseteq A$.

In order to establish the existence theorem of (WVIVIP), we first prove the following existence theorem of (SVIVIP).

THEOREM 3.2. *Assume that $f : X \to L(X, Y)$ and $g : K \to K$ are continuous, the set $\{y \in K : \langle f(x), y - g(x) \rangle \notin \mathrm{cl}D\}$ is convex or empty, $\forall x \in K$, and assume that $\langle f(x), x - g(x) \rangle \in \mathrm{cl}D, \forall x \in K$. If there exists a nonempty, compact, convex subset C of K, such that $\forall x \in K \backslash C, \exists\, y \in C$ such that*

$$\langle f(x), y - g(x) \rangle \notin \mathrm{cl}D,$$

then, (SVIVIP) has a solution. Furthermore, the solutions set of (SVIVIP) is closed.

Proof. Set $A = \{(x, y) \in K \times K : \langle f(x), y - g(x) \rangle \in \mathrm{cl}D\}$. The proof of the theorem consists of four steps.

Step 1. For each $x \in K, (x, x) \in A$, since $\langle f(x), x - g(x) \rangle \in \mathrm{cl}D, \forall x \in K$.

Step 2. $A_y = \{x \in K : (x, y) \in A\}$ is closed in K for all $y \in K$. In fact, let $\{x_\alpha\}$ be a net in A_y such that $x_\alpha \to x_0 \in K$. Since $x_\alpha \in A_y$, we know that $\langle f(x_\alpha), y - g(x_\alpha) \rangle \in \mathrm{cl}D$. Since $f : X \to L(X, Y)$ and $g : K \to K$ are continuous and $\mathrm{cl}D$ is closed, then $\langle f(x_0), y - g(x_0) \rangle \in \mathrm{cl}D$. It follows that $x_0 \in A_y$.

Step 3. From assumption, we have $A_x = \{y \in K : (x, y) \notin A\} = \{y \in K : \langle f(x), y - g(x) \rangle \notin \mathrm{cl}D\}$ is convex or empty, for each $x \in K$.

Step 4. Let $B = \{x \in K : (x, y) \in A, \forall y \in C\}$. We show that B is compact in C. By assumption, for each $x \in K \backslash C$, there exists $y \in C$ such that $\langle f(x), y - g(x) \rangle \notin \mathrm{cl}D$, that is, $(x, y) \notin A$, so that $x \notin B$. Thus, we have $B \subseteq C$. Since $B = \cap_{y \in C} A_y$, A_y is closed, and C is compact it follows that B is a closed, compact subset of C.

From the above four steps and Lemma 3.3, there exists $x^* \in K$ such that $\{x^*\} \times K \subseteq A$, that is, $\langle f(x^*), y - g(x^*) \rangle \in \mathrm{cl}D$ for all $y \in K$.

As step 2, we can prove that the solutions set of (SVIVIP) is closed. This completes the proof. □

EXAMPLE 3.2. Let $X = Y = R^2$, $K = R^2_+ = [0, \infty) \times [0, \infty)$, $C = [0, 1] \times [0, 1]$ and $D = (0, \infty) \times (0, \infty)$.

Let

$$g(x) = (x_2, x_1), \qquad f(x) \equiv f$$

and $\langle f(x), z \rangle = f(z) = (z_1 + z_2, z_1 + z_2)$ for any $x, z \in K$, with $x = (x_1, x_2)$ and $z = (z_1, z_2)$. Then,

$$\langle f(x), y - g(x) \rangle = ((y_1 + y_2) - (x_1 + x_2), (y_1 + y_2) - (x_1 + x_2))$$

for any $x, y \in K$, with $x = (x_1, x_2)$ and $y = (y_1, y_2)$, and all assumptions in Theorem 3.2 hold. It is easy to see that $(0, 0) \in K$ is a unique solution of (SVIVIP).

THEOREM 3.3. *Let D be an acute cone with apex at the origin and $0 \notin -D$. Let $f : X \to L(X, Y)$ and $g : K \to K$ be continuous. If all assumptions in Theorem 3.2 hold, then (WVIVIP) has a solution.*

Proof. It follows from Theorems 3.1(i) and 3.2 that the conclusion holds. This completes the proof. □

THEOREM 3.4. Let D be an acute cone with apex at the origin and $0 \notin -D$. *Let $f : X \to L(X, Y)$ and $g : K \to K$ be continuous. If* $\langle f(x^*), g(x^*) \rangle \in \mathrm{frt}(\mathrm{cl}D) \cup \mathrm{frt}(-\mathrm{cl}D)$ *for any solution x^* of (WVIVIP), and all assumptions in Theorem 3.3 hold, then (WVICP) has a solution.*

Proof. It follows from Theorems 3.1(ii) and 3.3 that the conclusion holds. This completes the proof. □

REMARK 3.4. If g is an identity mapping in Theorems 3.3 and 3.4, then we obtain the existence theorems of solutions for (VVIP) and (WVCP). Thus, we answered the open question proposed by Rapcsák [7].

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